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ON THE SOLUTIONS OF CERTAIN FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE n -TH ORDER

Abstract. The classes of solutions in $[0, \infty)$ of the general functional-differential equation (1) are studied. The equation (1) includes various types of functional-differential equations with deviated argument. The solutions are functions with discontinuous derivative of the n -th order.

In the earlier paper [2] the classes of solutions in $[0, \infty]$ of an abstract functional-differential equation of the form

$$(1) \quad \varphi^{(n)}(t) = \Theta F \varphi(t), \quad \Theta^2 = 1,$$

were studied. A solution φ of (1) was understood to be a function of the class $C^{(n)}$ in an interval $[a, b) \subset \mathbb{R}^1$.

Let us introduce certain spaces of functions:

Φ^n , $n = 0, 1, \dots$, denotes the space of functions $\varphi(t)$, $t \geq 0$, with continuous derivatives $\varphi^{(0)}, \varphi', \dots, \varphi^{(n)}$.

We write $\varphi(t) \geq \alpha$ ($\varphi(t) \leq \alpha$) if there exists a number $b \geq 0$ such that $\varphi(t) \geq \alpha$ ($\varphi(t) \leq \alpha$) for $t \geq b$ and $\varphi(t) \neq \alpha$ in any subinterval of $[0, \infty)$. Instead of $\varphi(t) \geq 0$ ($\varphi(t) \leq 0$) we write $S[\varphi] = 1$ ($S[\varphi] = -1$). As the limit $\lim \varphi(t)$ we always understand $\lim \varphi(t)$ as $t \rightarrow \infty$.

Ψ^n denotes the subspace of Φ^n consisting of functions φ such that $\varphi^{(k)}(t)$ have determined signs for sufficiently large t and $k = 0, 1, \dots, n$.

Ψ^{nk} denotes the subspace of Ψ^n containing functions φ satisfying the following conditions:

$$1^\circ \quad \varphi \in \Psi^n,$$

$$2^\circ \quad S[\varphi^{(i)}] S[\varphi] = 1 \text{ for } i = 0, 1, \dots, k,$$

$$3^\circ \quad S[\varphi^{(i)}] S[\varphi] = (-1)^{i-k} \text{ for } i = k+1, \dots, n-1, \text{ (when } k < n),$$

$$4^\circ \quad \lim \varphi^{(m)}(t) = 0 \text{ for } m = k+1, \dots, n-1 \text{ (when } k < n-1),$$

$$5^\circ \quad \lim \varphi^{(k)}(t) = g \neq \pm \infty \text{ exists and } g S[\varphi] \geq 0 \text{ (when } k \leq n-1).$$

\mathcal{B}^{nk} is the subspace of Ψ^{nk} consisting of functions φ for which $\lim \varphi^{(k)}(t) = 0$.

\mathcal{A}^n is the subspace of Φ^n consisting of functions φ for which

$$\sup \{t: \varphi(t) = 0\} = \infty.$$

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The following theorem was demonstrated in [2].

If

1° for any $\varphi \in \Phi^n$, $S[\varphi] = 1$ or $S[\varphi] = -1$ we have

$$S[\varphi] S[F\varphi] = 1$$

and

2° for any $\varphi \in \Phi^n$, $S[\varphi] = 1$ or $S[\varphi] = -1$, $|\varphi(t)| \geq c t^p$, $c = \text{const} > 0$, $p \in \mathbb{N}$, $0 \leq p < n-1$, we have

$$|\int_{-\infty}^{\infty} s^{n-p-2} F \varphi(s) ds| = \infty,$$

then all solutions of (1) which exist in $[0, \infty)$ belong to the following classes:

$\varphi^{(n)} = -F\varphi$	$\varphi^{(n)} = F\varphi$	
\mathcal{A}^n	$\mathcal{A}^n, \Psi^{nn}, \mathcal{B}^{n0}$	$n \text{ even}$
$\mathcal{A}^n, \mathcal{B}^{n0}$	\mathcal{A}^n, Ψ^{nn}	$n \text{ odd}$

The results formulated above will be extended to functional - differential equations of the form (1) but with a more general definition of a solution.

We define a solution φ of (1) as a real valued function which satisfies the following conditions:

1° φ is continuous for $t \geq 0$,

2° the derivatives $\varphi', \dots, \varphi^{(n-1)}$ exist for $t \geq 0$,

3° $\varphi^{(n)}(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of a sequence $\{t_1, \dots, t_m, \dots\} \subset [0, \infty)$ without any finite cluster point,

4° the right-hand derivatives $\varphi^{(n)}(t_j+)$ exist,

5° equation (1) is satisfied in every interval $[t_i, t_{i+1}) \subset [0, \infty)$.

REMARK 1. This type of generalization of a solution of (1) and $n = 1$ is necessary for a study of functional-differential equations which occur in mathematical models of certain biomedical phenomena ([6]).

REMARK 2. Some examples of equations of the form (1) for which the above generalization of a solution is useful are given below:

(a) $\varphi^{(n)}(t) = \Theta f(t, \varphi(t - E(\alpha(t))) + \Theta g(t, \varphi(t - \beta(t)))$,

(b) $\varphi^{(n)}(t) = p(t)f(\varphi(E(t)))$, $E(t) = \text{Entier}(t)$,

(c) $\varphi^{(n)}(t) = f(t, \varphi(t), \dots, \varphi^{(n-1)}(t), \varphi(t \pm \delta_0), \dots, \varphi^{(n-1)}(t \pm \delta_{n-1}), \varphi(E(t)), \dots,$

$$\varphi^{(n-1)}(E(t))) \cdot \int_{h(t)}^{k(t)} \varphi(t-s) d_s r(t, s),$$

$S[f] = \text{const}$, $h(t) \leq t \leq k(t)$, for fixed t the function r is non-decreasing in s .

Now we shall give some definitions and lemmas.

The expression $\varphi^{(n)}(t) \geq \alpha$ ($\varphi^{(n)}(t) \leq \alpha$) means here that $\varphi^{(n)}(t) \geq \alpha$ for b sufficiently large ($\varphi^{(n)}(t) \leq \alpha$ for $t \geq b$) except possibly a sequence $\{t_1, t_2, \dots\} \subset [0, \infty]$ for which $\varphi^{(n)}(t_i+) \geq \alpha$ ($\varphi^{(n)}(t_i+) \leq \alpha$). When $\alpha = 0$, then we say that $\varphi(t)$ is of constant sign in $[0, \infty)$. $S[\varphi] = 1$ or $S[\varphi] = -1$.

Let us define the following spaces of functions:

$\hat{\Phi}^n$, $n = 1, 2, \dots$, is the space of continuous functions $\varphi(t)$, $t \geq 0$, with continuous derivatives $\varphi', \dots, \varphi^{(n-1)}$ for $t \geq 0$. The derivative $\varphi^{(n)}$ exists for $t \geq 0$ possibly except for a sequence of points $\{t_1, t_2, \dots\} \subset [0, \infty)$ for which the right derivatives $\varphi^{(n)}(t_i +)$ exist.

$\hat{\Psi}^n$, $n = 0, 1, \dots$, is the subspace of $\hat{\Phi}^n$ with functions φ such that $\varphi^{(i)}(t)$, $i = 0, 1, \dots, n-1$ have determined signs for sufficiently large t and $\varphi^{(n)}(t) \geq '0$ or $\leq '0$.

$\hat{\Psi}^{nk}$ is defined by the conditions: $\varphi \in \hat{\Psi}^{nk}$ if and only if

1° $\varphi \in \hat{\Psi}^n$,

2° $S[\varphi^{(i)}]S[\varphi] = 1$ for $i = 0, 1, \dots, k$,

3° $S[\varphi^{(i)}]S[\varphi] = (-1)^{i-k}$ for $i = k+1, \dots, n-1$ (when $k < n$),

4° $\lim \varphi^{(m)}(t) = 0$ for $m = k+1, \dots, n-1$ (when $k < n-1$),

5° the limit $\lim \varphi^{(k)}(t) = g \neq \pm \infty$ exists and $gS[\varphi] \geq 0$ (when $k \leq n-1$).

$\hat{\Phi}^{nk}$ is the subspace of $\hat{\Psi}^{nk}$ consisting of functions φ for which $\lim \varphi(t) = 0$.

$\hat{\mathcal{A}}^n$ is the subspace of $\hat{\Phi}^n$ consisting of functions φ for which

$$\sup\{t: \varphi(t) = 0\} = \infty.$$

LEMMA 1. If $\varphi \in \hat{\Phi}^n$ and $\varphi^{(n)}(t) \geq '0$ ($\leq '0$) in the set $[0, \infty) \setminus \{t_1, t_2, \dots\}$, then $\varphi \in \hat{\Psi}^n$.

Proof. Let us consider the case $\varphi^{(n)}(t) \geq '0$. Then $\varphi^{(n)}(t) \geq 0$ for $t \geq b$, except possibly for the points $\{t_1, t_2, \dots\}$. From the formula

$$\varphi^{(n-1)}(t) = \varphi^{(n-1)}(a) + \int_a^t \varphi^{(n)}(s) ds$$

it follows that $\varphi^{(n-1)}(t)$, $t \geq a$, is monotonic and therefore of constant sign for sufficiently large t . The same is true for $\varphi^{(n-2)}(t), \dots, \varphi(t)$.

LEMMA 2. For every function $\varphi \in \hat{\Psi}^n$ there exists a number $b \geq 0$ and a natural number k , $0 \leq k \leq n$, such that $\varphi \in \hat{\Psi}^{nk}$. (In fact the class $\hat{\Psi}^n \subset \hat{\Phi}^n$, so when $\varphi \in \hat{\Psi}^n$ the functions $\varphi, \varphi', \dots, \varphi^{(n)}$ are of constant sign for $t \geq b$, when b is sufficiently large.)

Proof. The lemma is true for $n = 1$. Let us suppose that it is for $n-1$. At first, when $S[\varphi^{(n-1)}]S[\varphi^{(n-2)}] = -1$ and $\varphi^{(n-2)}(t) \geq '0$, ($\varphi^{(n-1)}(t) \leq '0$), then $\varphi^{(n)}(t)$ is of constant sign and the limit $\lim \varphi^{(n-1)}(t) = g \leq 0$ exists. When $g < 0$, then $\varphi^{(n-1)}(t) \leq 'c < 0$ and for b sufficiently large

$$0 < \varphi^{(n-2)}(t) < \varphi^{(n-2)}(b) - cb + ct,$$

which is impossible. So we have $\varphi^{(n-1)}(t) \leq '0$, $\lim \varphi^{(n-1)}(t) = g = 0$, from which it follows that $\varphi^{(n)}(t) \geq '0$. We see that in this case the signs of $\varphi^{(n-2)}$, $\varphi^{(n-1)}$, $\varphi^{(n)}$ alternate. The existence of the integer k follows from our assumption on $k-1$.

Now let us discuss the case $S[\varphi^{(n-1)}]S[\varphi^{(n-2)}] = 1$. We can take $k = n$, when $S[\varphi^{(n)}]S[\varphi^{(n-1)}] = 1$, and $k = n-1$ when $S[\varphi^{(n)}]S[\varphi^{(n-1)}] = -1$ and the lemma is again true.

LEMMA 3. When $\varphi(t) \in \hat{\Psi}^{nk}$, $0 \leq k \leq n$, $\varphi^{(n)}(t) \leq 0$ and $f(t)$ is a continuous and non-negative function in the interval $[b, \infty)$ such that for $b \leq t \leq v$

$$S[\varphi^{(n)}] \int_t^v \varphi^{(n)}(s) ds \geq \int_t^v f(s) ds,$$

then

$$(2) \quad S[\varphi^{(\kappa)}] \varphi^{(\kappa)}(t) \geq \frac{1}{(n-\kappa-1)!} \int_t^\infty (s-t)^{n-\kappa-1} f(s) ds,$$

$$t \geq b, \kappa = n-1, n-2, \dots, k.$$

The last integral is convergent and

$$S[\varphi^{(\kappa)}] = (-1)^{n-\kappa} S[\varphi^{(n)}].$$

Proof. From the assumptions of the lemma it follows that $\varphi^{(n-1)} \geq 0$ and

$$-\int_t^v \varphi^{(n)}(s) ds = -\varphi^{(n-1)}(v) + \varphi^{(n-1)}(t) \geq \int_t^v f(s) ds.$$

So we have

$$\varphi^{(n-1)}(t) \geq \int_t^v f(s) ds.$$

The last formula is a particular case of (2) with $\kappa = n-1$. Let us suppose that (2) is true for $\kappa > k$. First we discuss the case $S[\varphi^{(\kappa)}] = 1$ and, for the induction proof, assume that the formula

$$\varphi^{(\kappa)}(t) \geq \frac{1}{(n-\kappa-1)!} \int_t^\infty (s-t)^{n-\kappa-1} f(s) ds$$

is true. By integration in the interval $[t, v]$ we get

$$\begin{aligned} \varphi^{(\kappa-1)}(v) - \varphi^{(\kappa-1)}(t) &= \frac{1}{(n-\kappa-1)!} \int_t^v \varphi^{(\kappa)}(s) ds \\ &\geq \frac{1}{(n-\kappa-1)!} \int_t^v \left(\int_s^\infty (u-s)^{n-\kappa-1} f(u) du \right) ds. \end{aligned}$$

But $S[\varphi^{(\kappa-1)}] = -1$ for $\varphi \in \tilde{\Psi}^{nk}$ and

$$\varphi^{(\kappa-1)}(t) \geq \frac{1}{(n-\kappa-1)!} \int_t^v \left(\int_s^\infty (u-s)^{n-\kappa-1} f(u) du \right) ds \geq 0.$$

So we have

$$\varphi^{(\kappa-1)}(t) \geq \frac{1}{(n-\kappa-1)!} \int_t^\infty \left(\int_s^\infty (u-s)^{n-\kappa-1} f(u) du \right) ds \geq 0.$$

Let us estimate the integral

$$\begin{aligned} \lim_{v \rightarrow \infty} \int_t^v \left(\int_s^\infty (u-s)^{n-\kappa-1} f(u) du \right) ds &= \lim_{v \rightarrow \infty} \int_s^\infty \left(\int_t^v (u-s)^{n-\kappa-1} f(u) ds \right) du \\ &= \lim_{v \rightarrow \infty} \int_u^\infty f(u) \left(\int_t^\infty (u-s)^{n-\kappa-1} ds \right) du = \lim_{v \rightarrow \infty} \int_u^\infty f(u) \frac{(u-t)^{n-\kappa} - (u-v)^{n-\kappa}}{n-\kappa} du \\ &\geq \int_u^\infty \frac{(u-t)^{n-\kappa}}{n-\kappa} f(u) du. \end{aligned}$$

This finishes the induction. The case $S[\varphi^{(\kappa)}] = -1$ is similar.

Now we impose some hypotheses for the operation F in (1).

The operation F is in the space $\hat{\Phi}^n$ with values in the same space, $n \geq 1$.

Hypothesis H_1 . For $\varphi \in \hat{\Phi}^n$, $S[\varphi]S[F\varphi] = 1$.

Hypothesis H_2 . When $\varphi \in \hat{\Phi}^n$, $S[\varphi] = 1$ or $S[\varphi] = -1$, $|\varphi(t)| \geq c t^p$, $c = \text{const} > 0$, $p \in \mathbb{N}$, $0 \leq p < n-1$, then

$$\int_0^\infty |s^{n-p-2} F\varphi(s)| ds = \infty.$$

THEOREM 1. When the hypothesis H_1 is true, then every solution φ of equation (1) which exists in the interval $[b, \infty)$ belongs to one of the classes \mathcal{A}^n or $\hat{\Psi}^{nk}$ i.e.

if $\varphi^{(n)} = -F\varphi$, then $\varphi \in \mathcal{A}^n$ or $\varphi \in \hat{\Psi}^{nk}$, $0 \leq k < n$,

if $\varphi^{(n)} = F\varphi$, then $\varphi \in \mathcal{A}^n$ or $\varphi \in \hat{\Psi}^{nk}$, $0 \leq k \leq n$.

Proof. Suppose that $\varphi \notin \mathcal{A}^n$. Then $S[\varphi] = 1$ or $S[\varphi] = -1$ and from Lemmas 1 and 2 it follows that $\varphi \in \hat{\Psi}^{nk}$, $0 \leq k \leq n$.

THEOREM 2. When the hypotheses H_1 and H_2 are satisfied, then every solution φ of equation (1) which exists in the interval $[b, \infty)$ belongs to one of the classes \mathcal{A}^n , $\hat{\Psi}^{nn}$, $\hat{\mathcal{B}}^{n0}$, i.e.

$\varphi^{(n)} = -F\varphi$	$\varphi^{(n)} = F\varphi$	
\mathcal{A}^n	$\mathcal{A}^n, \hat{\Psi}^{nn}, \hat{\mathcal{B}}^{n0}$	n even
$\mathcal{A}^n, \hat{\mathcal{B}}^{n0}$	$\mathcal{A}^n, \hat{\Psi}^{nn}$	n odd

Proof. Let us consider a solution φ of equation (1) in the interval $[b, \infty)$, such that $\varphi \notin \mathcal{A}^n$. From Theorem 1 it follows that $\varphi \in \hat{\Psi}^{nk}$, $0 \leq k \leq n$. It is sufficient to demonstrate that the index k is equal to zero. When $\varphi \in \hat{\Psi}^{nk}$ then $S[\varphi^{(k)}] = 1$ or $S[\varphi^{(k)}] = -1$. Suppose that $S[\varphi^{(k)}] = 1$, $\varphi^{(k)} \geq 0$ and $\varphi^{(k-1)}(t)$ is positive and non-decreasing. There exists an $\alpha > 0$ such that $\varphi^{(k-1)} \geq 2\alpha$. The last inequality gives $\varphi(t) \geq t^{k-1}$ for $t \geq b$ and b sufficiently large. Let us take $f(t) = F\varphi(t)$. From the form of equation (1) and the hypothesis H_2 , for $p = k-1$ we get

$$\int_0^\infty s^{n-k-1} f(s) ds = \infty.$$

Integrating "per partes" the integral

$$\int_\tau^t s^{n-k-1} \varphi^{(n)}(s) ds$$

we get

$$\int_\tau^t s^{n-k-1} \varphi^{(n)}(s) ds = t^{n-k-1} \varphi^{(n-1)}(t) - (n-k-1)(n-k-2)t^{n-k-2}$$

$$\cdot \varphi^{(n-2)}(t) + \dots \pm (n-k-1)! \varphi^{(k)}(t) + \int_\tau^t \varphi^{(k)}(s) ds + C$$

$$> (n-k-1)! \varphi^{(k)}(t) + C = C + \int_\tau^t (s-\tau)^{n-k-1} \varphi^{(n)}(s) ds.$$

For $t \rightarrow \infty$ these inequalities lead to a contradiction $\infty < \infty$ and hence our assumptions about $k: k \geq 1$ is false. A similar contradiction is obtained when $S[\varphi^{(k)}] = -1$.

Suppose now that $\varphi(t) \geq \alpha > 0$. From the hypothesis H_2 , with $p = 0$, it follows that

$$(3) \quad \infty = \int s^{n-2} F\varphi(s) ds < \int s^{n-1} F\varphi(s) ds.$$

But from Lemma 3 and $f(t) = \alpha > 0$, $\kappa = 0$ it follows that

$$\varphi(t) \geq \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} \alpha ds \geq 0.$$

From the last inequality and Lemma 3 we get

$$(4) \quad \infty > \frac{1}{(n-1)!} \int s^{n-1} F\varphi(s) ds.$$

Conditions (3) and (4) leads to

$$\infty = \int s^{n-2} F\varphi(s) ds < \int s^{n-1} F\varphi(s) ds < \infty.$$

This contradiction finishes the demonstration.

REMARK. Equation (1) is a generalization of a great number of functional-differential equations with or without deviation of the argument. The formulation of Theorem 2 is very general. The theorem includes not only the classical results of W. B. Fite [4], J. G. Mikusiński [5], A. Bielecki and T. Dłotko [2], T. Dłotko [3], but also the newest results reported by B. G. Zhang and N. Parhi [6], A. R. Aftabizadeh and J. Wiener [1], B. G. Zhang [7].

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